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LETTER TO THE EDITOR

Renormalisability and renormalisation of the 'true' self-avoiding random walks

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Abstract. Renormalisation of the recently proposed field-theoretic model for the 'true' self-avoiding random walk is analysed. Apart from a special case, which corresponds to the problem of a random walk in a random environment, the model is shown to be non-renormalisable in the case of short-range interactions, contrary to earlier conjectures. The long-range version of this field-theoretic model is shown to be renormalisable in certain regions of the parameter space, and renormalisation group analysis of the model is carried out to two-loop order in the region corresponding to the 'true' self-avoiding random walk with long-range repulsion. In a different region of the parameter space the anomalous dimension of the diffusion coefficient is determined exactly in perturbation theory.

It has been argued that the recently proposed [1] model of the 'true' self-avoiding random walk (TSAW) gives rise to a renormalisable field theory with short-range interactions, to which renormalisation group (RG) techniques have been applied [2] to study the long-time behaviour of TSAW. The long-range interaction version of this field theory has also been analysed in a similar fashion [3]. In this letter we show that in earlier work an infinite set of marginal operators has been overlooked, the account of which renders the field theory, in general, non-renormalisable. More explicitly, the field-theoretic model [2] is characterised by three coupling constants g_1, g_2 and g_3, and we show that, in the space of these three parameters, the model is renormalisable only on the line $g_1=g_3$, $g_2=0$, where its asymptotic behaviour coincides with that of the model of a random walk in a random-velocity field [4]. The long-range version of this model [3] also turns out to be non-renormalisable for arbitrary coupling constants. However, it is renormalisable in the plane $g_2=0$, and we have carried out to two-loop order the RG analysis of the long-range TSAW (to which corresponds the line $g_2=g_3=0$). We also show that on the line $g_1=g_2=0$ the anomalous dimension of the diffusion coefficient is determined exactly by the fixed-point equation of the RG.

In the continuum limit, TSAW is defined by the equations [2]

$$\mathrm{d}\boldsymbol{R}(t)/\mathrm{d}t=-g_1\nabla\rho(\boldsymbol{R}(t),t)+\eta(t) \tag{1}$$

$$\partial\rho(\boldsymbol{x},t)/\partial t=\delta(\boldsymbol{x}-\boldsymbol{R}(t)) \tag{2}$$

where $\boldsymbol{R}$ is the position of the random walker, and η is a Gaussian noise with zero mean and correlation function $\overline{\eta_m(t)\eta_n(t')}=2D_0\delta(t'-t)\delta_{mn}$, where D_0 is the 'bare'

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diffusion coefficient. These equations describe the short-range TSAW, whereas for the long-range version the equation (1) is replaced by [3]

$$\mathrm{d}\boldsymbol{R}(t)/\mathrm{d}t = -g_1 \nabla\phi(\boldsymbol{R}(t), t) + \eta(t) \tag{3}$$

where the function ϕ is related to ρ by

$$\phi(\boldsymbol{x}, t) = \int \mathrm{d}\boldsymbol{x}' K(\boldsymbol{x}-\boldsymbol{x}')\rho(\boldsymbol{x}', t) \tag{4}$$

with the kernel $K(\boldsymbol{x})$ given by

$$K(\boldsymbol{x}) = \int \frac{\mathrm{d}\boldsymbol{q}}{(2\pi)^d} \frac{\exp(\mathrm{i}\boldsymbol{q}\boldsymbol{x})}{\boldsymbol{q}^{2\alpha}}. \tag{5}$$

We will be interested in the probability distribution $P(\boldsymbol{x}, t)$ of the random walks started at the origin

$$P(\boldsymbol{x}, t) = \overline{\delta(\boldsymbol{x}-\boldsymbol{R}(t))}$$

where $\boldsymbol{R}(t)$ is the solution of the problem (1), (2) with the initial condition $\boldsymbol{R}(0) = 0$. For the retarded Green function

$$G(\boldsymbol{x}, t) = \theta(t)P(\boldsymbol{x}, t)$$

a functional integral representation may be constructed [2]

$$G(\boldsymbol{x}-\boldsymbol{x}', t-t') = \langle\varphi(\boldsymbol{x}, t)\tilde{\varphi}(\boldsymbol{x}', t')\rangle = \int \mathrm{D}\varphi\, \mathrm{D}\tilde{\varphi}\, \varphi(\boldsymbol{x}, t)\tilde{\varphi}(\boldsymbol{x}', t') \exp(S)$$

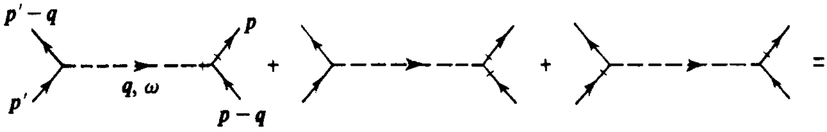
where the action S is of the form

$$S = \int \mathrm{d}t\, \mathrm{d}\boldsymbol{x}\, \tilde{\varphi}(\boldsymbol{x}, t)[-\partial_t + D_0\nabla^2]\varphi(\boldsymbol{x}, t)$$
$$-g_1 \int \mathrm{d}t\, \mathrm{d}t'\, \mathrm{d}\boldsymbol{x}\, \varphi(\boldsymbol{x}, t)\nabla\tilde{\varphi}(\boldsymbol{x}, t)\theta(t-t')\nabla[\varphi(\boldsymbol{x}, t')\tilde{\varphi}(\boldsymbol{x}, t')]. \tag{6}$$

It is not difficult to see that the upper critical dimension of this field theory is two, and that the form of the interaction in (6) is not preserved under renormalisation: one-loop calculations [2] show that at least one additional four-point term (the corresponding coupling constant is denoted by g_2) is generated by renormalisation and the possibility that a third term (with g_3) appears in higher orders cannot be ruled out. This leads to the interaction S_{int} of the form

$$S_{\mathrm{int}} = \int \mathrm{d}t\, \mathrm{d}t'\, \mathrm{d}\boldsymbol{x}\{-g_1\varphi(\boldsymbol{x}, t)\nabla\tilde{\varphi}(\boldsymbol{x}, t)\theta(t-t')\nabla[\varphi(\boldsymbol{x}, t')\tilde{\varphi}(\boldsymbol{x}, t')]$$
$$+ g_2\nabla\varphi(\boldsymbol{x}, t)\nabla\tilde{\varphi}(\boldsymbol{x}, t)\theta(t-t')\varphi(\boldsymbol{x}, t')\tilde{\varphi}(\boldsymbol{x}, t')$$
$$+ g_3\varphi(\boldsymbol{x}, t)\nabla\tilde{\varphi}(\boldsymbol{x}, t)\theta(t-t')\nabla\varphi(\boldsymbol{x}, t')\tilde{\varphi}(\boldsymbol{x}, t')\}. \tag{7}$$

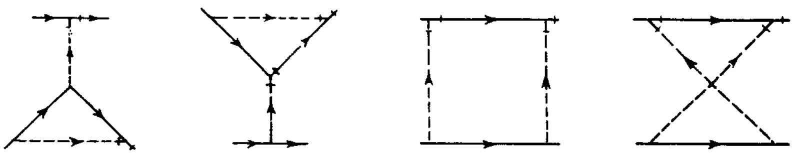
We denote the $\varphi\tilde{\varphi}$ propagator of this field theory by a full line, to which corresponds the factor $(-\mathrm{i}\omega + D_0\boldsymbol{q}^2)^{-1}$, and in which the arrow points from the $\tilde{\varphi}$ end to the φ end (it also shows the direction of time). The notation for the interaction vertices is depicted in figure 1, where the slashes on full lines correspond to the derivatives in the interaction (7), and the arrow on the broken line shows the direction of time.



$$\frac{1}{-\mathrm{i}\omega+\mathrm{O}}(-g_1 pq + g_2 p(p-q) + g_3 pp')$$

Figure 1. Retarded four-point vertex structures, the arrows show the direction of time, and the slashes on full lines correspond to the derivatives at the interaction vertices (7). The direction of the momentum flow is chosen according to the arrows. In the effective 'static' graphs, from which the renormalisation constants are extracted, the θ function factor $(-\mathrm{i}\omega+\mathrm{o})^{-1}$ is replaced by unity, whereas the momentum structure of the vertices is the same as shown here.

The RG analysis of this field theory has been carried out at one-loop level [2], but, unfortunately, it is not sufficient to consider only the four-point interaction (7), as we shall now show. Consider the one-loop graphs of figure 2, which contribute to the four-point vertex renormalisation (for brevity, we have shown graphs containing the g_1 vertex only). The relevant contribution of each graph to the vertex renormalisation constants is given by an effective 'static' graph with momentum integration only, in which the θ function factors of broken lines are replaced by unity and the internal frequencies are set equal to zero. The crucial point is that g_2 vertices may be attached in arbitrary number to these graphs without any change in the dimensionality of the corresponding momentum integral: if the g_2 vertex is added in such a way that the derivatives at the end of the broken line act to internal lines, then the corresponding momenta dimensionally compensate for the additional propagator, and the large-momentum behaviour of the integrand is the same as in the original graph. This is illustrated by the examples of figure 3. In general, the g_2 vertex is inevitably generated by renormalisation, which thus gives rise to an infinite set of divergences of different types, i.e. the field theory (6), (7) is not renormalisable. There are, however, two cases, in which the g_2 vertex is not generated by renormalisation, if it is absent in the initial model. First, if we choose $g_1 = g_2 = 0$, $g_3 \neq 0$, then it may readily be checked that this model is self-consistent in the sense that four-point vertex structures corresponding to g_1 and g_2 are absent in the perturbation expansion. Nevertheless, this model also turns out to be non-renormalisable, and the reason is that it contains another set of divergent 1PI (one-particle irreducible) graphs with an arbitrary number of external φ and $\tilde{\varphi}$ legs. Examples of these are shown in figure 4. Second, if we choose $g_2 = 0$, $g_1 = g_3 \neq 0$,



(*a*) (*b*) (*c*) (*d*)

Figure 2. One-loop graphs of the short-range TSAW field theory (6), (7), which renormalise the four-point interaction term. For brevity, we have depicted graphs with g_1 vertex only. These graphs give rise also to the vertex with g_2, and for the general case (7) similar graphs with all vertex structures (and thus with differently arranged slashes) must be taken into account.

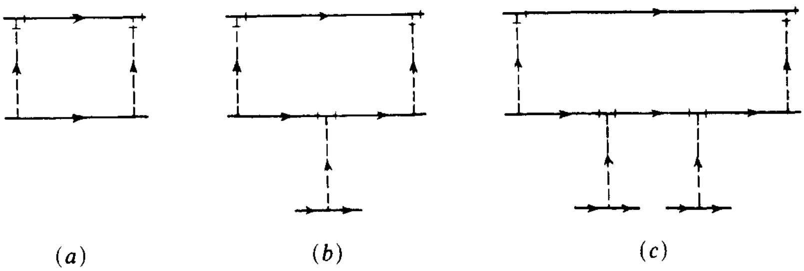


Figure 3. Examples of the structure of divergent one-loop graphs from the 1PI graphs of figure 2: (*a*) is the original four-point graph and (*b*) and (*c*) are, respectively, six and eight point graphs obtained by attaching g_2 vertices to it.

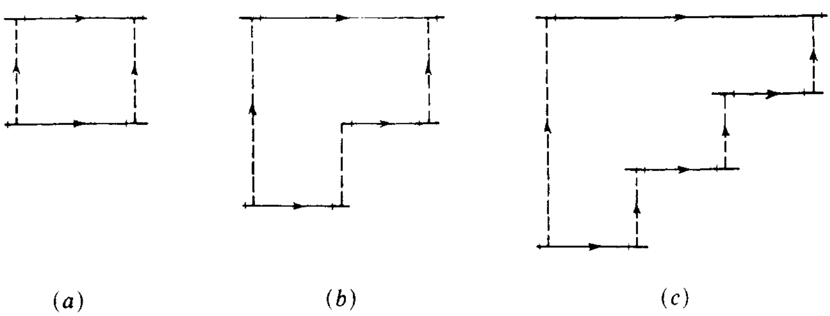


Figure 4. Examples of divergent one-loop graphs of the short-range model (7) in the case $g_1=g_2=0$, $g_3\neq 0$. Graphs (*a*), (*b*) and (*c*) yield divergent contributions to four, six and eight point 1PI Green functions, respectively, and it is clear from these examples that there are similar contributions to Green functions of arbitrary order.

the interaction (7) takes the form

$$S_{\text{int}}=-g_1\int \mathrm{d}t\,\mathrm{d}t'\,\mathrm{d}x\,\varphi(x,t)\nabla\tilde{\varphi}(x,t)\theta(t-t')\varphi(x,t')\nabla\tilde{\varphi}(x,t') \tag{8}$$

which, apart from the retarded character of the interaction, coincides with the interaction of the field-theoretic model of a random walk in a random environment ('random random walk') [4]. The essential feature of the interaction (8) is that the $\tilde{\varphi}$ field enters only with a derivative. In the graphs of perturbation theory, the corresponding momenta therefore factorise at the vertices with external $\tilde{\varphi}$ legs, thus rendering all graphs with more than two external $\tilde{\varphi}$ legs superficially convergent (i.e. they yield only finite contributions to renormalisation constants after subtraction of divergences corresponding to divergent subgraphs). As a result, the model with the interaction (8) is multiplicatively renormalisable with the same long-time asymptotic behaviour as in the model of a random walk in a random-velocity field [4]. From these arguments it follows that, apart from the special case $g_1=g_3$, $g_2=0$, the previous analyses [2] of this model are not sufficient to determine the correct long-time behaviour for $d\leq 2$.

The situation is, however, different in the case of long-range TSAW. In the formalism of two fields φ and $\tilde{\varphi}$ the kernel (5) would lead to interactions non-local in space, and to avoid them we introduce auxiliary scalar fields ψ, $\tilde{\psi}$, and vector fields A, $\tilde{A}$ and express the retarded Green function G of the long-range version of the model (6), (7) in the form

$$G(x-x', t-t') = \int \mathrm{D}\varphi\, \mathrm{D}\tilde{\varphi}\, \mathrm{D}\psi\, \mathrm{D}\tilde{\psi}\, \mathrm{D}A\, \mathrm{D}\tilde{A}\, \varphi(x, t)\tilde{\varphi}(x', t') \exp(S)$$

with the action S given by

$$\begin{aligned} S = -\int & \mathrm{d}t\, \mathrm{d}x\, \mathrm{d}x' [\tilde{A}(x, t) K^{-1}(x-x')\partial_t A(x', t) + \tilde{\psi}(x, t) K^{-1}(x-x')\partial_t \psi(x', t)] \\ & + \int \mathrm{d}t\, \mathrm{d}x [\tilde{\varphi}(x, t)(-\partial_t + D_0\nabla^2)\varphi(x, t) - g_1 \nabla\psi(x, t)\varphi(x, t)\nabla\tilde{\varphi}(x, t) \\ & + g_2 \psi(x, t)\nabla\varphi(x, t)\nabla\tilde{\varphi}(x, t) + \tilde{\psi}(x, t)\varphi(x, t)\tilde{\varphi}(x, t) \\ & + g_3 A(x, t)\varphi(x, t)\nabla\tilde{\varphi}(x, t) + \tilde{A}(x, t)\tilde{\varphi}(x, t)\nabla\varphi(x, t)] \end{aligned} \tag{9}$$

where the kernel K is defined by the relation (5), and we have used the same notation for the coupling constants as in the short-range case (7). The field theory for the long-range TSAW problem (2)-(5) corresponds to the choice $g_1 \neq 0$, $g_2 = g_3 = 0$ in the action (9). The elements of the perturbation theory for this model are essentially the same as in the short-range case, apart from that the broken lines instead of the mere θ-function factor (i.e. $1/(-\mathrm{i}\omega + \mathrm{o})$) contain a non-trivial momentum dependence: $1/[(-\mathrm{i}\omega + \mathrm{o})q^{2\alpha}]$. Power counting shows that the upper critical dimension d_c is now equal to $d_c = 2 + 2\alpha$, and—the most remarkable difference from the short-range case—the 'bad' one-particle irreducible (1PI) four-point graphs of figure 2 are convergent for finite $\alpha > 0$. It is not difficult to see that all 1PI graphs, which have more than two external φ or $\tilde{\varphi}$ legs, do not contain superficial divergences in this case, and therefore they are not relevant in the RG sense. However, three-point graphs with external φ and $\tilde{\varphi}$ legs (examples of which are the triangular 1PI subgraphs in figure 2) still contain, in general, superficial divergences, and they give rise to the renormalisation of the interaction vertices in (9). It should be noted that the structure of the three-point interaction vertices in the action (9) is such that it is preserved under renormalisation for each coupling constant separately: e.g., if we set $g_1 \neq 0$, and $g_2 = g_3 = 0$, then vertices corresponding to the coupling constants g_2 and g_3 are not generated by renormalisation. This holds for all three cases of a single non-vanishing coupling constant, and it implies, in particular, that the field-theoretic version of the long-range TSAW [3] (with $g_1 \neq 0$ only) is multiplicatively renormalisable, whereas in the short-range case the interaction corresponding to g_2 appears with the subsequent infinite set of marginal operators. It is not difficult to see, however, that the same problem is present also in the general long-range model (9) when $g_2 \neq 0$: if we attach g_2 vertices to logarithmically divergent three-point graphs this does not change the large-momentum behaviour of the corresponding loop integral and we are again faced with the problem of generating an infinite set of marginal operators. Therefore, we conclude that even in the long-range case the field theory (9) is renormalisable if and only if $g_2 = 0$.

We have carried out one-loop calculations for the general long-range model (9), and for the renormalisable case $g_2 = 0$ our results confirm the results of Peliti and Zhang [3]. However, our results for the case $g_2 \neq 0$ are different from these, but we do not quote them here, since they do not give the complete information about the asymptotic behaviour of the general model. Instead, we present the results of a two-loop calculation of the long-range TSAW ($g_1 \neq 0$, $g_2 = g_3 = 0$), which reveal singular behaviour of the long-range model near $\alpha = 0$.

It is convenient to introduce a new coupling constant $u_{1,0}=g_1 D_0^{-2} C_\alpha$, where $C_\alpha = [2^{1+2\alpha}\pi^{1+\alpha}\Gamma(1+\alpha)]^{-1}$. We write the basic action [6] (which contains renormalised parameters only) in the form

$$S_B = -\int dt\, dx\, dx'\, \tilde{\psi}(x,t) K^{-1}(x-x')\partial_t \psi(x',t) + \int dt\, dx[\tilde{\varphi}(x,t)(-\partial_t + D\nabla^2)\varphi(x,t)$$
$$- u_1 \mu^\varepsilon D^2 C_\alpha^{-1} \nabla\psi(x,t)\varphi(x,t)\nabla\tilde{\psi}(x,t) + \tilde{\psi}(x,t)\varphi(x,t)\tilde{\varphi}(x,t)] \quad (10)$$

where μ is the scale-setting parameter, and $\varepsilon = 2+2\alpha-d$. The renormalised parameters of this action are related to the bare ones by

$$D_0 = Z_D D \qquad u_{1,0} = u_1 \mu^\varepsilon Z_1 Z_D^{-2} \quad (11)$$

where Z_D and Z_1 are, respectively, the renormalisation constants of the diffusion coefficient, and the first interaction vertex of the action (10) (the second vertex is not renormalised). We use dimensional regularisation with minimal subtractions, and to two-loop order calculations with (10) yield

$$Z_D = 1 - \frac{u_1}{(1+\alpha)\varepsilon} + \frac{u_1^2}{4(1+\alpha)^2\varepsilon^2}\left(-1 + \frac{2+5\alpha+2\alpha^2}{2\alpha(1+\alpha)}\varepsilon\right) \quad (12)$$

$$Z_1 = 1 - \frac{u_1}{2(1+\alpha)\varepsilon} + \frac{u_1^2}{8(1+\alpha)^2\varepsilon^2}\left(-2 + \frac{2+4\alpha+\alpha^2}{\alpha(1+\alpha)}\varepsilon\right). \quad (13)$$

Note the poles at $\alpha=0$ in the second-order terms. They appear due to the fact that two-loop graphs contain the 1PI four-point graphs of figure 2 as subgraphs, which are convergent for finite α, but become divergent in the limit $\alpha \to 0$. This shows in the relations (12) and (13) in the form of poles $1/\alpha$. An analogous situation arises in the long-range random walk problem, for which it has been shown [5] that the results of the long-range model hold for $\alpha > \alpha^* = O(\varepsilon)$. A similar analysis is possible also for the model (10), but we do not dwell on it here. From (11)-(13) we obtain the following expression for the beta function

$$\beta_1(u_1) = \mu \frac{\partial u_1}{\partial \mu}\bigg|_0 = u_1\left[-\varepsilon + \frac{3}{2(1+\alpha)} u_1 - \frac{2+6\alpha+3\alpha^2}{4\alpha(1+\alpha)^3} u_1^2\right] \quad (14)$$

where, and henceforth, the subscript 0 indicates that the derivative is taken at fixed values of the bare parameters (11). The non-trivial fixed point

$$u_1^* = \tfrac{2}{3}(1+\alpha)\varepsilon\left(1 + \frac{2+6\alpha+3\alpha^2}{9\alpha(1+\alpha)}\varepsilon + O(\varepsilon^2)\right) \quad (15)$$

is perturbatively infrared stable, and thus controls the long-distance and long-time behaviour of the model.

The anomalous dimension of the diffusion coefficient is the value of the function

$$\gamma_D = \mu \frac{\partial \ln D}{\partial \mu}\bigg|_0 \quad (16)$$

at the fixed point of RG. For $d < d_c = 2+2\alpha$, we obtain

$$\gamma_D^* = \gamma_D(u_1 = u_1^*) = -\tfrac{2}{3}\varepsilon + \frac{2+3\alpha}{27\alpha(1+\alpha)}\varepsilon^2$$

which, through the relation $z=2+\gamma_D^*$, determines the dynamic exponent z and thus the asymptotic behaviour of the model at the long-time limit. For the mean-square displacement of the random walker we obtain

$$\overline{R^2(t)} \sim t^{2/(2+\gamma_D^*)} \sim t^{1+\varepsilon/3-(2-3\alpha-6\alpha^2)\varepsilon^2/[54\alpha(1+\alpha)]}$$

which implies superdiffusive behaviour (note that the first-order expressions for the exponent $\nu=1/(2+\gamma_D^*)$ for both the long-range TSAW and the random walk problem given in [3] are not correct). At the upper critical dimension we obtain logarithmic enhancement of diffusion

$$\overline{R^2(t)} \sim t(\ln t)^{2/3}.$$

Finally, we would like to point out that in the other renormalisable case with a single coupling constant ($g_1=g_2=0$, $g_3\neq 0$) of the long-range model (9), the anomalous dimension γ_D^* of the diffusion coefficient may be calculated exactly in the perturbation theory. In this case, only the last two terms in the interaction (9) are present, and it is not difficult to see that momenta, which correspond to the derivatives in the interaction terms, factorise in three-point 1PI graphs at the vertices with external φ and $\tilde{\varphi}$ legs. This renders the remaining loop integrals superficially convergent. In the minimal subtraction scheme the corresponding renormalisation constant is therefore trivial: $Z_3=1$. From (16) and the definitions $u_{3,0}=g_3D_0^{-2}C_\alpha$, $u_{3,0}=u_3\mu^\varepsilon Z_3Z_D^{-2}$ we obtain

$$\beta_3(u_3)=\mu\frac{\partial u_3}{\partial\mu}\bigg|_0=u_3\Big(-\varepsilon-2\gamma_D(u_3)\Big).$$

From this relation it follows that the fixed-point equation $\beta_3(u_3^*)=0$ determines the anomalous dimension $\gamma_D^*=\gamma_D(u_3=u_3^*)$ to all orders in perturbation theory. For the anomalous asymptotic behaviour of the mean-square displacement we obtain

$$\overline{R^2(t)} \sim t^{2/(2-\varepsilon/2)} \qquad d<d_c=2+2\alpha \tag{17}$$

or

$$\overline{R^2(t)} \sim t(\ln t)^{1/2} \qquad d=d_c=2+2\alpha$$

which are the same as in the model of a random walk in a transverse random-velocity field with long-range correlations [5]. It is also interesting to note that the relation (17) corresponds to a value of the exponent ν which is exactly the same as was predicted in the long-range TSAW problem (2)-(5) from a simple Flory argument [3].

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